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Multiplication Rules for Schur and Quasisymmetric Schur Functions

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MULTIPLICATION RULES FOR SCHUR AND QUASISYMMETRIC SCHUR FUNCTIONS

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ABSTRACT

An important problem in algebraic combinatorics is finding expansions of products of symmetric functions as sums of symmetric functions. Schur functions form a well-known basis for the ring of symmetric functions. The Littlewood-Richardson rule was introduced to expand the product of two Schur functions as a positive sum of Schur functions. Remmel and Whitney introduced an algorithmic way to find the coefficients of Schur functions appearing in the expansion. Haglund et al. introduced quasisymmetric Schur functions as a refinement of Schur functions. For quasisymmetric Schur functions, the Littlewood-Richardson rule was introduced to expand the product of a Schur and quasisymmetric Schur function as the positive sum of quasisymmetric Schur functions. We determine an algorithm similar to the Remmel-Whitney rule to find the coefficients of quasisymmetric Schur functions appearing in the expansion.

CHAPTER 1

INTRODUCTION

Symmetric functions are functions that are invariant under permutation of variables. The collection of symmetric functions forms a ring. Schur functions are a basis for this ring [6]. There are connections between Schur functions and the representation theory of $\mathfrak{S}_n$ [6].

Quasisymmetric Schur functions were introduced by Haglund et al. [1] as a refinement of Schur functions, and they form a basis for the ring of quasisymmetric functions. Quasisymmetric Schur functions share many combinatorial properties of Schur functions [1, 2, 3].

The product of two or more Schur functions can be written as a positive sum of Schur functions. Identifying which functions appear and their coefficients can be challenging and can be done using the Littlewood-Richardson rule [4].

There is an algorithm that, when given two Schur functions, produces a rooted tree where each of the leaves is a standard Young tableau. The shape of each such tableau is the index of a Schur function appearing in the expansion of the product. This algorithm is called the Remmel-Whitney rule [5]. There is a bijection that establishes the equivalence of the Littlewood-Richardson rule and the Remmel-Whitney rule [5].

Similar to the algorithm by Remmel and Whitney for the product of two Schur functions, we found an algorithm for multiplying a Schur function by a quasisymmetric Schur function.

Section 2 covers definitions related to symmetric functions and Schur functions. Section 3 contains the Remmel-Whitney rule along with related definitions and theorems. In Section 4, we define the quasisymmetric Schur function. Section 5 contains information on the Littlewood-Richardson rule for quasisymmetric Schur functions, along with an algorithm for multiplying a particular Schur function and a quasisymmetric Schur function.

CHAPTER 2

BACKGROUND

This section covers the background information on Schur functions, including definitions and a proof that Schur functions are symmetric. There are three traditional ways to define Schur functions; as a determinant, as a quotient of alternants, or as a generating function for semistandard Young tableaux [6]. We take the third approach in this paper.

Definition 1. An *integer partition* of n is a nonincreasing sequence of positive integers which sum to n. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n. Then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\sum \lambda_i = n$. Each λ_i is called a part of λ. The size of λ is n and is denoted $|\lambda|$. The *Ferrers diagram* of λ is a collection of left-justified boxes where row i contains λ_i boxes and row 1 is the bottom row, following the French convention. We denote the cell in the i^{th} row and j^{th} column by (i, j).

For example, one partition of 7 is $\lambda = (5, 2)$ with the Ferrers diagram as seen in Figure 2.1.

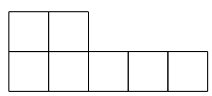


Figure 2.1: The Ferrers diagram of $(5, 2)$.

Definition 2. A *semistandard Young tableau* (SSYT) of shape λ is a filling of the Ferrers diagram of a partition of n with positive integer entries such that each row is weakly increasing from left to right and each column is strictly increasing from bottom to top. The set of all SSYT of shape λ is represented by SSYT(λ).

One element of SSYT$(5, 2)$ is shown in Figure 2.2.

2	3			
1	1	1	2	3

Figure 2.2: An element of SSYT$(5, 2)$.

Definition 3. A *standard Young tableau* (SYT) is a filling of the Ferrers diagram of a partition of n with distinct entries $1, \ldots, n$ each used exactly once such that each row is strictly increasing from left to right and each column is strictly increasing from bottom to top.

One element of SYT $(5, 3, 2)$ is shown in Figure 2.3.

4	9			
3	6	10		
1	2	5	7	8

Figure 2.3: An element of SYT $(5, 3, 2)$.

Definition 4. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be partitions with $\mu_i \leq \lambda_i$ for all i. A *skew partition* (λ/μ) is obtained by taking the diagram of λ and deleting the cells of shape μ from left to right. The result is called the *skew diagram* of λ/μ. The size of λ/μ is denoted by $|\lambda/\mu| = |\lambda| - |\mu|$.

The diagram for $(4, 3, 2)/(2, 1)$ is shown in Figure 2.4.

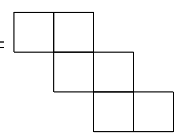
$\lambda/\mu =$ 

Figure 2.4: The diagram for $(4, 3, 2)/(2, 1)$.

Definition 5. A *skew tableau* of shape λ/μ is obtained by inserting positive integer entries into each cell. The tableau is *semistandard* if the filling is weakly increasing from left to right and strictly increasing up columns. The tableau is *standard* if each row is strictly increasing from left to right and each column is strictly increasing from bottom to top and the entries $1, \ldots, |\lambda/\mu|$ are each used exactly once.

A skew standard tableau of shape $(4, 3, 2)/(2, 1)$ is shown in Figure 2.5.

Definition 6. The *content monomial*, also known as the *weight*, of a tableau T of shape λ is the monomial $x^T = \prod_{i=1}^{n} x_i^{v_i}$, where v_i is the number of times the label i appears in T.

$$\begin{array}{|c|c|cc}
\hline
5 & 6 & & \\
\hline
 & 3 & \multicolumn{1}{|c|}{4} & \\
\cline{2-4}
 & & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\
\cline{3-4}
\end{array}$$

Figure 2.5: A standard skew tableau for $(4,3,2)/(2,1)$.

The content monomial for the tableau in Figure 2.2 is $x_1^3 x_2^2 x_3^2$.

Definition 7. Let λ be a partition. The *Schur function* S_λ is

$$S_\lambda(x_1, x_2, x_3, \ldots) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T,$$

the sum of all content monomials for all $T \in \mathrm{SSYT}(\lambda)$.

Most often we consider a finite collection of variables $x_1, \ldots, x_n$. In Figure 2.6 we see the eight tableau in SSYT$(2,1)$ filled with 1's, 2's, and 3's and the sum of the content monomials of each tableaux in SSYT$(2,1)$ resulting in $S_{(2,1)}(x_1, x_2, x_3)$.

$$\begin{array}{cccc}
\begin{array}{|c|c|} \hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} &
\begin{array}{|c|c|} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} &
\begin{array}{|c|c|} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} &
\begin{array}{|c|c|} \hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} \\
x_1x_2x_3 & x_1x_2^2 & x_1^2x_2 & x_1^2x_3 \\
\begin{array}{|c|c|} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} &
\begin{array}{|c|c|} \hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline \end{array} &
\begin{array}{|c|c|} \hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline \end{array} &
\begin{array}{|c|c|} \hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} \\
x_1x_2x_3 & x_2^2x_3 & x_2x_3^2 & x_1x_3^2
\end{array}$$

$$S_{(2,1)}(x_1, x_2, x_3) = 2x_1x_2x_3 + x_1x_2^2 + x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_2x_3^2 + x_1x_3^2$$

Figure 2.6: The Schur function $S_{(2,1)}(x_1, x_2, x_3)$.

Definition 8. A *symmetric function* is a function $f(x_1, \ldots, x_n)$ such that for every permutation $\sigma \in \mathfrak{S}_n$, $\sigma f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$, where $\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. This $\mathfrak{S}_n$-action permutes the subscripts of the x_i's.

The function $f_1(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_1x_3$ is symmetric. However, the function $f_2(x_1, x_2, x_3) = x_1^2x_2 + x_2x_3 + x_1x_3^3$ is not symmetric since $(1,2)f_2 = x_2^2x_1 + x_1x_3 + x_2x_3^3 \neq f_2$.

Lemma 1. *Let λ be a partition. Then S_λ is a symmetric function.*

Proof. Let λ be a partition and $\sigma \in \mathfrak{S}_n$. We need to show $\sigma S_\lambda = S_\lambda$. Since every permutation σ can be written as a product of adjacent transpositions $(i \;\; i+1)$, we need to only prove $(i \;\; i+1)S_\lambda = S_\lambda$. We do this by showing for some $F \in \mathrm{SSYT}(\lambda)$ containing j entries of i and k entries of $i+1$ there exists some $F' \in \mathrm{SSYT}(\lambda)$ with k entries of i and j entries of $i+1$.

Suppose $F \in \mathrm{SSYT}(\lambda)$ contains the entry i appearing j times and the entry $i+1$ appearing k times. We now produce F' by swapping labels i and $i+1$ in a particular way. If i and $i+1$ are in the same column, they are fixed in order to preserve the column increasing condition for a semistandard tableau. We call all other entries of i or $i+1$ *free*. Observe that the free i's are all to the right of the fixed i's in that row, an the free $i+1$'s are all to the left of the fixed $i+1$'s. So all free i's and $i+1$'s are consecutive in each row. In each row, we replace the m free entries of i and l free entries of $i+1$ with l entries of i and m entries of $i+1$. In that row, m entries of i's and l entries of $i+1$'s need to be replaced in increasing order. Observe that this is an involution, meaning $(i\ i+1)[(i\ i+1)](F) = F$, so this produces a bijection between tableaux with content monomials containing the terms $x_i^{a_i} x_{i+1}^{a_{i+1}}$ and $x_i^{a_{i+1}} x_{i+1}^{a_i}$. The resulting tableau is $F' \in \mathrm{SSYT}(\lambda)$. Therefore, $(i \;\; i+1)S_\lambda = S_\lambda$. So, S_λ is a symmetric function. $\square$

Figure 2.7 shows the bijection used to show Schur functions are symmetric. Entries 4 and 5 are fixed if they appear in the same column and all other 4's and 5's are free. Row by row replace i free 4's and j free 5's with j free 4's and i free 5's.

$$
\begin{array}{|c|c|c|c|c|c|c|}
\cline{1-1} 5 \\
\cline{1-2} 4 & 4 \\
\cline{1-4} 3 & 3 & 5 & 5 \\
\cline{1-5} 2 & 2 & 3 & 4 & 5 \\
\hline 1 & 1 & 1 & 1 & 2 & 5 & 5 \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{|c|c|c|c|c|c|c|}
\cline{1-1} 5 \\
\cline{1-2} 4 & 5 \\
\cline{1-4} 3 & 3 & 4 & 5 \\
\cline{1-5} 2 & 2 & 3 & 4 & 4 \\
\hline 1 & 1 & 1 & 1 & 2 & 4 & 4 \\
\hline
\end{array}
$$

Figure 2.7: An example of bijection given in Lemma 1 where $(i \;\; i+1) = (4, 5)$.

CHAPTER 3

THE REMMEL-WHITNEY RULE

The object of this section is to multiply two Schur functions together and identify what coefficients appear in the Schur function expansion of their product. The Remmel-Whitney Rule [5] provides an algorithmic way to find the coefficients of each Schur function appearing in $S_\lambda \cdot S_\mu$. The same result can be obtained by using the Littlewood-Richardson rule [4].

To state the algorithm, we first need some background definitions. We then state the Remmel-Whitney algorithm, the Littlewood-Richardson rule, and show the bijection establishing their equivalence.

Definition 9. Given a skew partition λ/μ, the *reading word* of a tableau in SSYT(λ/μ) is obtained by reading the entries from right to left by starting with the bottom row.

The reading word for the skew shape $(4,3,2)/(2,1)$ in Figure 2.5 is 214365.

Definition 10. Given a partition λ of n, the *reverse lexicographic filling* of λ, denoted $R(\lambda)$, is the unique filling of λ with reading word $123\cdots n$.

Definition 11. The operation $\lambda * \mu$ is defined as the skew diagram of $(\lambda_1+\mu_1, \lambda_2+\mu_1, \ldots, \lambda_k+\mu_1, \mu_1, \ldots, \mu_m)/(\mu_1)^k$ where $(\mu_1)^k$ contains k parts of size μ_1.

The skew diagram of skew shape $(3,2)*(2,2,1)$ and $R(\lambda*\mu)$ for $\lambda=(3,2)$ and $\mu=(2,2,1)$ can be seen in Figure 3.1.

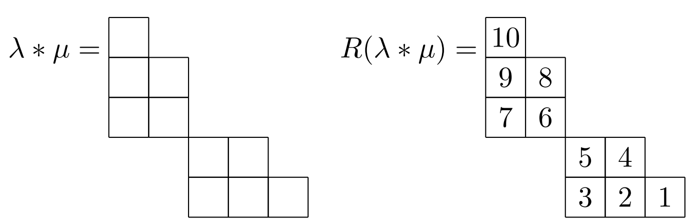


Figure 3.1: The skew diagram of $(3,2)*(2,2,1)$ and R((3,2) * (2,2,1)).

Definition 12 (Remmel, Whitney [5])**.** The set $\mathcal{O}(\lambda*\mu)$ is the set of tableaux T satisfying:

1. If x and $x+1$ occur in the same row of the reverse lexicographic filling of $\lambda * \mu$, then $x+1$ must occur weakly below and strictly to the right of the cell containing x in T.

2. In the reverse lexicographic filling of $\lambda * \mu$, if y occurs in the same column and in a row above the cell with the entry x, then y must be strictly above and weakly to the left of x in T.

We denote by $\mathcal{O}_\nu(\lambda * \mu)$ the tableaux in $\mathcal{O}(\lambda * \mu)$ of shape ν.

By looking at the reverse lexicographic filling of $(3,2) * (2,2,1)$, we verify that the tableaux in Figure 3.1 are in $\mathcal{O}((3,2) * (2,2,1))$. The two tableaux have distinct shapes.

10			
8			
6			
4	5	9	
1	2	3	7

10		
8	9	
6	7	
4	5	
1	2	3

Figure 3.2: Two tableaux in $\mathcal{O}((3,2) * (2,2,1))$.

We now detail the algorithm used to generate the elements of $\mathcal{O}(\lambda * \mu)$.

Algorithm 2 (Remmel-Whitney rule [5]). *To generate the rooted tree whose leaves are elements of $\mathcal{O}(\lambda * \mu)$:*

1. *Start by using the reverse lexicographic filling of the skew-diagram of $\lambda * \mu$. For $|\lambda| = n$ and $|\mu| = m$, the reverse lexicographic filling of $\lambda * \mu$ will contain entries $1, \ldots, |\lambda| + |\mu| = n + m$.*

2. *Place 1 in a cell. This is the root of the tree.*

3. *For $i = 2, \ldots, n+m$, do the following at each vertex v in level $i-1$: Place i in all possible ways following the restrictions given in Definition 12. Each distinct placement creates a filling that becomes a new vertex in the tree, with an edge connected to v.*

4. *Once all entries $1, \ldots, n+m$ have been placed, the leaves are the fillings in $\mathcal{O}(\lambda * \mu)$. To obtain the summands occuring in the product $S_\lambda \cdot S_\mu$, we need to replace each leaf of the tree with the Schur function of the same shape. So, $S_\lambda \cdot S_\mu$ is the sum of the tableaux*

$T \in \mathcal{O}(\lambda * \mu)$, given as

$$S_\lambda \cdot S_\mu = \sum_{T \in \mathcal{O}(\lambda * \mu)} S_{sh(T)},$$

where $sh(T)$ is the shape of T.

Consider the reverse lexicographic filling of $\lambda * \mu = (2,1) * (1)$ as seen in Figure 3.3. By starting with the entry 1, we need to look for all positions where the entry 2 can be placed. We need to apply the first rule in the Definition 12 since 1 and 2 appear in the same row. So, 2 must be placed weakly below and strictly to the right of 1. There is only one placement for the 2, which is directly to the right of 1 as seen in Figure 3.3.

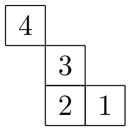
Now we need to find all possible positions for the 3. Since the 3 is in the same column and strictly above the 2, then by the second condition in Definition 12, the entry 3 must be placed strictly above the 2 and weakly to the left. In order to satisfy the conditions for standard Young tableaux, the 3 can only be placed above the 1 as seen in Figure 3.3.

Lastly, we need to find all possible positions for the entry 4. Since neither condition in Definition 12 apply to the position of 4 in the lexicographic filling, then the 4 can be placed in all positions satisfying the conditions for standard Young tableaux. We then replace each leaf of the tree with the Schur function of the same shape as seen in Figure 3.3. For a larger example of Algorithm 2, see Figure 3.6.

Theorem 3. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n. Applying the Remmel-Whitney rule to $R(\lambda)$ yields the standard Young tableaux of shape λ, $T(\lambda)$, with the entries $1, \ldots, \lambda$, in row 1, $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ in row 2, and $\lambda_1 + \cdots + \lambda_{i-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_i$ in row i.*

Proof. Assume $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n. In the Remmel-Whitney rule, we first place 1 in a cell as the root of the tree. Let $T_i(\lambda)$ be the result of placing entries $1, \ldots, i$ using the Remmel-Whitney rule and assume $T_i(\lambda)$ has entries $1, \ldots, \lambda_1$ in row 1 and $\lambda_1 + 1, \ldots, i$ in row 2. Now we use the Remmel-Whitney rule to show $T_{i+1}(\lambda)$ satisfies the desired condition.

If $i+1$, in some row j, is in the same row of $R(\lambda)$ as i and $i+1$ is directly above some y, then $i+1$ must be placed strictly right and weakly below i in $T_i(\lambda)$ and strictly above and weakly left of y in $T_i(\lambda)$ by Definition 12. So, $i+1$ must be placed in the same row as i. Since $\lambda_j \leq \lambda_{j-1}$,



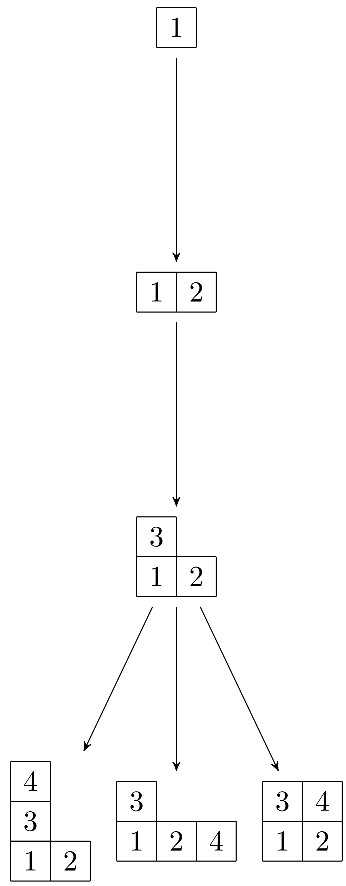


Figure 3.3: The tree and Schur function expansion for $S_{(1)} \cdot S_{(2,1)}$.

$$S_{(1)}S_{(2,1)} = S_{(2,2)} + S_{(2,1,1)} + S_{(3,1)}$$

then the placement of $i+1$ is weakly left of y.

Now we need to consider if i and $i+1$ are not in the same row in $R(\lambda)$. Then $i+1$ is above some entry y in $R(\lambda)$. By Definition 12, $i+1$ must appear weakly left and strictly above y. So $i+1$ must start a new row.

Therefore, applying the Remmel-Whitney rule to $R(\lambda)$ yields the standard Young tableaux of shape λ, $T(\lambda)$, with the entries $1, \ldots, \lambda$ in row 1, $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ in row 2, and $\lambda_1 + \cdots + \lambda_{j-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_k$ in row k. $\square$

We now state the Littlewood-Richardson rule following the permutation in [6], and then conclude by showing that the Remmel-Whitney rule and the Littlewood-Richardson rule are equivalent.

Definition 13. A *lattice word*, $u = u_1, \ldots, u_n$, is a sequence of integers such that for each j, $1 \leq j \leq n$ and every $i \geq 1$ in the sequence, the number of times $i+1$ appears in $u = u_1, \ldots, u_j$ is less than or equal to the number of i's that appear in $u = u_1, \ldots, u_j$. A lattice word always starts with 1.

For example, 1211223113 is a lattice word and 122133321 is not a lattice word, because the number of 3's in the first seven characters is larger than the number of 2's.

Definition 14. A *Littlewood-Richardson tableau* of shape ν/λ and content μ is a semistandard Young tableau T of shape ν/λ with content monomial $x^T = x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$ and the reading word of T is a lattice word.

We denote by $LR_\mu(\nu/\lambda)$ the set of *Littlewood-Richardson tableaux* of shape ν/λ with content monomial $x^{\mu_1} x^{\mu_2} \cdots x^{\mu_k}$ where $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$.

Theorem 4 (Littlewood-Richardson [4])**.** *The Littlewood-Richardson rule for $S_\lambda \cdot S_\mu$ is defined as:*

$$S_\lambda \cdot S_\mu = \sum_{\substack{\nu \\ |\nu| = |\lambda| + |\mu|}} g^\nu_{\lambda,\mu} S_\nu$$

such that $g^\nu_{\lambda,\mu}$ is the number of tableaux $T \in LR_\mu(\nu/\lambda)$ of shape ν/λ where the reading word of T is a lattice word and $x^T = x_1^{\mu_1} \cdots x_n^{\mu_n}$.

A full proof of Theorem 4 took over 40 years [6].

$$F_1 = \begin{array}{|c|c|cc} \hline 2 & 3 & \multicolumn{1}{c}{} & \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c|}{2} & \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c|}{1} \\ \cline{3-4} \end{array} \qquad F_2 = \begin{array}{|c|c|cc} \hline 1 & 3 & \multicolumn{1}{c}{} & \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c|}{2} & \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c|}{1} \\ \cline{3-4} \end{array}$$

Figure 3.4: Littlewood-Richardson tableaux of shape $(4,3,2)/(2,1)$ and content $(3,2,1)$.

For example, given $\nu = (4,3,2), \lambda = (2,1)$, and $\mu = (3,2,1)$, we need to fill the the tableaux of shape ν/λ such that the reading word is a lattice word and the requirements of a semistandard tableau are met. The two tableaux that meet these requirements are seen in the Littlewood-Richardson tableaux in Figure 3.4 and have reading words 112132 and 112231. Therefore, the coefficient $g^{\nu}_{\lambda,\mu}$ is 2.

We now show the equivalence of these two rules.

Theorem 5 (Remmel, Whitney[5]). *Let* $\mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n)$*, and* $\lambda = (\lambda_1, \ldots, \lambda_p)$ *such that* $|\nu| = |\lambda| + |\mu|$*. Then,* $g^{\nu}_{\lambda,\mu}$ *is the number of* $Q \in \mathcal{O}_{\nu}(\lambda * \mu)$*.*

Proof. We need to show that there is a bijection $\phi : \mathcal{O}_{\nu}(\lambda * \mu) \to LR_{\mu}(\nu/\lambda)$. Start with a tableau $Q \in \mathcal{O}_{\nu}(\lambda * \mu)$. Delete $T(\lambda)$ from Q. For each cell remaining, if the entry x appears in row i of μ in the reverse lexicographic filling of $\lambda * \mu$, replace x with i. Since the rows in the diagram of μ are weakly decreasing from bottom to top, then the number of i's is greater than or equal to the number of $i + 1$'s. The tableau is in the form of $LR_{\mu}(\nu/\lambda)$ where $u(\phi(Q))$ is a lattice word and $x^{\phi(Q)} = x_1^{\mu_1} \cdots x_n^{\mu_n}$.

We need to show that there is a bijection ψ such that $\psi : LR_{\mu}(\nu/\lambda) \to \mathcal{O}_{\nu}(\lambda * \mu)$. Start with $F \in LR_{\mu}(\nu/\lambda)$, where $u(F)$ is a lattice word and $x^F = x_1^{\mu_1} \cdots x_n^{\mu_n}$. For each entry in F, replace all entries i in F in reverse reading order by the entries in row i of μ in $R(\lambda * \mu)$ from smallest to largest. This ensures the resulting tableau satisfies the conditions from Definition 12. Finally, insert $T(\lambda)$ into the skewed cells to obtain $\psi(F)$. Then $\psi(F) \in \mathcal{O}_{\nu}(\lambda * \mu)$. Observe that $(\phi \circ \psi)(F) = F$, $(\psi \circ \phi)(Q) = Q$, and thus ϕ is a bijection with inverse ψ.

□

For example, suppose $\nu = (4, 3, 2), \mu = (3, 2, 1)$, and $\lambda = (2, 1)$. We want to use the bijection in order to obtain tableaux in $\mathcal{O}_{(4,3,2)}((2, 1) * (3, 2, 1))$. Figure 3.4 shows the two possible Littlewood-Richardson tableaux of shape $(4, 3, 2)/(2, 1)$ with content $(3, 2, 1)$. For each entry in F_1 and F_2, replace all entries i in F_1 and F_2 in reverse reading order by the entries in row i of $(3, 2, 1)$ in $R((2, 1) * (3, 2, 1))$ from smallest to largest. Finally, insert $T((2, 1))$ into the skewed cells to obtain $\psi(F_1)$ and $\psi(F_2)$ as seen in Figure 3.5.

Now start with the tableaux in Figure 3.5, we need to delete $T((2, 1))$ from Q_1 and Q_2. For each cell remaining, if the entry x appears in row i of $(3, 2, 1)$ in the reverse lexicographic filling of $(2, 1) * (3, 2, 1)$, replace x with i. Since the rows in the diagram of $(3, 2, 1)$ are weakly decreasing from bottom to top, then the number of i's is greater than or equal to the number of $i+1$'s. The tableaux is in the form of $LR_{(3,2,1)}((4,3,2)/(2,1))$ where $u(\phi(Q_1))$ and $u(\phi(Q_2))$ are lattice words and $x^{\phi(Q_1)} = x^{\phi(Q_2)} = x_1^3 x_2^2 x_3$ as seen in Figure 3.4.

$$\begin{array}{ccccc} 9 & & & & \\ 8 & 7 & & & \\ 6 & 5 & 4 & & \\ & & & 3 & \\ & & & 2 & 1 \end{array} \qquad Q_1 = \begin{array}{cccc} 7 & 9 & & \\ 3 & 4 & 8 & \\ 1 & 2 & 5 & 6 \end{array} \qquad Q_2 = \begin{array}{cccc} 4 & 9 & & \\ 3 & 7 & 8 & \\ 1 & 2 & 5 & 6 \end{array}$$

Figure 3.5: The reverse lexicographic filling of $(2,1)*(3,2,1)$ and elements of $\mathcal{O}_{(4,3,2)}((2,1)*(3,2,1))$.

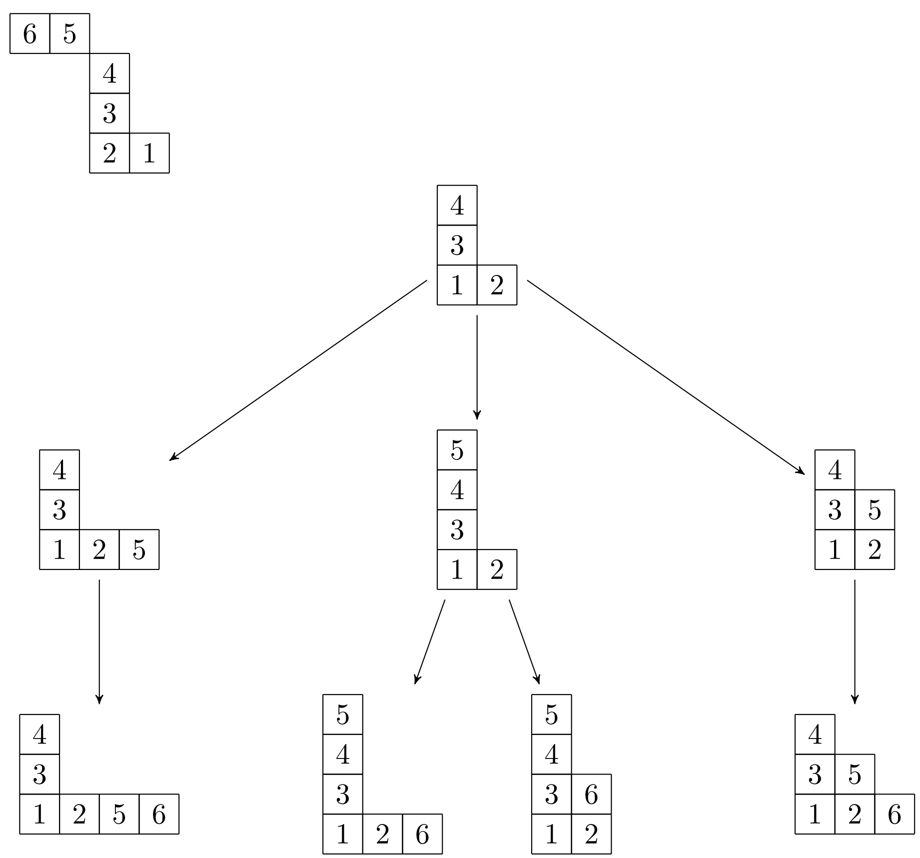


Figure 3.6: An example of using the algorithm to find the expansion of $S_{(2)}S_{(2,1,1)}$.

$$S_{(2)}S_{(2,1,1)} = S_{(3,2,1)} + S_{(4,1,1)} + S_{(2,2,1,1)} + S_{(3,1,1,1)}$$

CHAPTER 4

QUASISYMMETRIC SCHUR FUNCTIONS

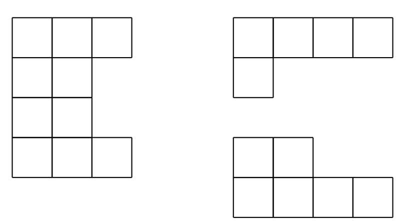
Quasisymmetric Schur functions were defined by Haglund et al. [1] as a refinement of Schur functions, meaning each Schur function can be written as a positive sum of quasisymmetric Schur functions. Quasisymmetric Schur functions retain many of the properties of Schur functions, such as Robinson-Schensted-Knuth and Littlewood-Richardson rules [2, 3].

In order to define quasisymmetric Schur functions, we first need to define compositions and quasisymmetric functions. We then define the objects that generate quasisymmetric Schur functions and conclude by showing quasisymmetric Schur functions refine the Schur functions.

Definition 15. A *composition* α of n, denoted $\alpha \models n$, is a sequence of positive integers whose sum is n. A *weak composition* α of n is a sequence of nonnegative integers whose sum is n and parts of size 0 are allowed.

Definition 16. Given a (possibly weak) composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, the *diagram* of α is the collection of left-justified boxes with α_i cells in row i, where row 1 is the bottom row.

A composition of 10 is $(3, 2, 2, 3)$ and a weak composition of 11 is $(4, 1, 0, 2, 4)$ with diagrams seen in Figure 4.1.



$(3, 2, 2, 3)$ $(4, 1, 0, 2, 4)$

Figure 4.1: The diagram of the composition $(3, 2, 2, 3)$ and the weak composition $(4, 1, 0, 2, 4)$.

Definition 17. The *partition underlying a composition* α is the partition obtained by placing each part of α in weakly decreasing order to obtain the partition denoted $\boldsymbol{\lambda}(\alpha)$.

The diagram of the partition underlying the composition $\alpha = (3, 1, 2, 1, 3)$, which is $\boldsymbol{\lambda}(\alpha) = (3, 3, 2, 1, 1)$, can be seen in Figure 4.2.

$\alpha = (3, 1, 2, 1, 3)$ $\qquad$ $\boldsymbol{\lambda}(3, 1, 2, 1, 3) = (3, 3, 2, 1, 1)$

Figure 4.2: The partition underlying the composition $(3, 1, 2, 1, 3)$.

Definition 18. A function $f(x_1, \ldots, x_n)$ is a *quasisymmetric function* if and only if for all compositions $\alpha = (\alpha_1, \ldots, \alpha_s)$ and $1 \le i_1 < i_2 < \cdots < i_s \le N$ where $s \le N$, the coefficient of $\prod_{j=1}^{s} x_{i_j}^{\alpha_j}$ is the same as the coefficient of $\prod_{j=1}^{s} x_j^{\alpha_j}$.

A quasisymmetric function in three variables is $f_1(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3$. The function $f_1(x_1, x_2, x_3)$ is not symmetric since $(1, 2)f_1 = x_2^2 x_1 + x_1 2^2 x_3 + x_1^2 x_3 \neq f_1$. Not all quasisymmetric functions are symmetric, but all symmetric functions are quasisymmetric. The function $f_2(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$ is both symmetric and quasisymmetric.

We are now in a position to define the objects that generate the quasisymmetric Schur functions.

Definition 19 (Haglund et al. [1])**.** A *semistandard Young composition tableau* T is a filling of $\alpha = (\alpha_1, \ldots, \alpha_l)$ where the entries in the cells must follow three conditions. Let $m = \max\{\alpha_1, \ldots, \alpha_l\}$. Create a rectangular filling, $\hat{T}$, from T by setting $\hat{T}(i, k) = T(i, k)$ when $(i, k) \in \alpha$ and $\hat{T}(i, k) = \infty$ when $(i, k) \notin \alpha$ and $i \le l$, $k \le m$.

1. The entries in each row of $\hat{T}$ must be weakly increasing from left to right.

2. The entries in the first column $\hat{T}$ must be strictly increasing from bottom to top.

3. The following triple condition must be met. For $j < i$, $k < m$, and

$a = \hat{T}(j, k+1), b = \hat{T}(i, k+1)$, and $c = \hat{T}(i, k)$, if $a \neq \infty$ and $a \geq c$ then $a > b$. See Figure 4.3 for the triple configuration.

$$\begin{array}{|c|c|}
\hline
c & b \\
\hline
\end{array}$$

$$\begin{array}{cc}
 & \boxed{a}
\end{array}$$

Figure 4.3: Triples in a semistandard Young composition tableau.

We denote the set of semistandard Young composition tableaux of a given shape α as $\mathrm{SSYCT}(\alpha)$.

An example of an element of $\mathrm{SSYCT}(1, 2, 1)$ can be seen in Figure 4.4.

$$T = \begin{array}{|c|c}
\cline{1-1}
3 & \\
\hline
2 & \multicolumn{1}{c|}{2} \\
\hline
1 & \\
\cline{1-1}
\end{array}$$

$$\hat{T} = \begin{array}{|c|c|}
\hline
3 & \infty \\
\hline
2 & 2 \\
\hline
1 & \infty \\
\hline
\end{array}$$

Figure 4.4: One element of $\mathrm{SSYCT}(1, 2, 1)$.

Lemma 6 (Haglund et al. [1])**.** *Given $T \in \mathrm{SSYCT}(\alpha)$, the entries in a column of T are all distinct.*

Proof. Since column 1 of T is strictly increasing by definition, then the entries of column 1 of T are all distinct.

Now we need to show all other entries in a column of T are distinct. Suppose there are two cells in column k with entries $a = b$, a appearing below b. Consider a triple of cells consisting of the entries a, b and c where c is to the left of b as in Figure 4.3. By Definition 19, since rows are weakly increasing, if $a < c$, then $a < b$, but $a = b$, so we must have $a \geq c$. Since $a \geq c$ and $a = b$, then a, b, c do not satisfy the triple rule. Therefore, all entries in a column of T are distinct. $\square$

The definition of a quasisymmetric Schur function is similar to the definition of a Schur function. The quasisymmetric Schur functions are generated by the weights of the semistandard

Young composition tableaux of a given shape. We formally define quasisymmetric Schur functions and show how Schur functions can be decomposed into quasisymmetric Schur functions.

Definition 20 (Haglund et al. [1])**.** Let α be a composition. The quasisymmetric Schur function indexed by α is

$$QS_\alpha = \sum_{T \in \mathrm{SSYCT}(\alpha)} x^T.$$

Consider when $\alpha = (1,2,1)$ and $n = 4$. Figure 4.5 shows the five tableaux in $\mathrm{SSYCT}(1,2,1)$ filled with 1's, 2's, 3's, and 4's and the sum of the content monomials of each tableau in $\mathrm{SSYCT}(1,2,1)$, which is $QS_{(1,2,1)}(x_1, x_2, x_3, x_4)$.

$$\begin{array}{ccccc}
\begin{array}{|c|c|}\hline 4 & \\ \hline 2 & 3 \\ \hline 1 & \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 3 & \\ \hline 2 & 2 \\ \hline 1 & \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 4 & \\ \hline 2 & 2 \\ \hline 1 & \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 4 & \\ \hline 3 & 3 \\ \hline 1 & \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 4 & \\ \hline 3 & 3 \\ \hline 2 & \\ \hline\end{array} \\
x_1x_2x_3x_4 & x_1x_2^2x_3 & x_1x_2^2x_4 & x_1x_3^2x_4 & x_2x_3^2x_4
\end{array}$$

$$QS_{(1,2,1)}(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 + x_1x_2^2x_3 + x_1x_2^2x_4 + x_1x_3^2x_4 + x_2x_3^2x_4$$

Figure 4.5: The SSYCT of shape $(1,2,1)$ generating $QS_{(1,2,1)}(x_1, x_2, x_3, x_4)$.

Theorem 7 (Haglund et al. [1])**.** *Let λ be a partition. Then,*

$$S_\lambda = \sum_{\boldsymbol{\lambda}(\alpha) = \lambda} QS_\alpha.$$

Proof. We need to show that there is a bijection ρ from $\mathrm{SSYT}(\lambda)$ to $\bigcup_{\alpha : \boldsymbol{\lambda}(\alpha)=\lambda} \mathrm{SSYCT}(\alpha)$. We need to first show that for every $T \in \mathrm{SSYT}(\lambda)$, there is a rearrangement of λ that results in a composition $\rho(T) \in \mathrm{SSYCT}(\alpha)$ for some α with $\boldsymbol{\lambda}(\alpha) = \lambda$. We will start by taking the first column of $T \in \mathrm{SSYT}(\lambda)$ and keep it the same in $\rho(T)$. To place the entries of the second column of T into the second column of $\rho(T)$, we will scan down the first column of $\rho(T)$ and place each entry of the second column of T in the highest row possible so that each row of $\rho(T)$ is weakly increasing, starting with the smallest entry of the second column first. Continue likewise for remaining columns.

Now we need to show that the triple conditions are satisfied by $\rho(T)$. Consider the triple of cells in Figure 4.3. We need to show if $a \geq c$, then $a > b$. Assume $a \geq c$ and $a < b$. Since the entries in each column of T must be strictly increasing, we know $a \neq b$. Since $a < b$, a must be inserted into $\rho(T)$ before b. Since a was not inserted next to c, then $a < c$. This is a contradiction, so the triple rules are satisfied. For an example of ρ, see Figure 4.6.

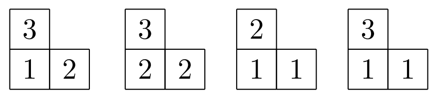
Let $F \in \bigcup_{\alpha:\boldsymbol{\lambda}(\alpha)=\lambda} \mathrm{SSYCT}(\alpha)$. Define $\gamma(F)$ in the following way. Place the entries from column k in F in column k of $\gamma(F)$ so that columns increase from bottom to top. Since the entries in each column of F were distinct, the columns of $\gamma(F)$ are strictly increasing. Since the rows of F are weakly increasing, then the rows in the new tableau are also weakly increasing. Observe that $\gamma(\rho(T)) = T$ for all $T \in \mathrm{SSYT}(\lambda)$. Therefore, $\gamma(F) \in \mathrm{SSYT}(\lambda)$.

We need to show ρ is injective. Suppose $\rho(T) = \rho(S)$. The entries in each column of $\rho(T)$ and $\rho(S)$ are the same. Thus, in T and S, the column entries are the same. Therefore, $T = S$. $\square$

$$\underset{T}{\begin{array}{cccc} 5 & & & \\ 4 & 4 & & \\ 2 & 3 & 4 & 4 \\ 1 & 1 & 2 & 2 \end{array}} \quad \longrightarrow \quad \underset{\rho(T)}{\begin{array}{cccc} 5 & & & \\ 4 & 4 & 4 & 4 \\ 2 & 3 & & \\ 1 & 1 & 2 & 2 \end{array}}$$

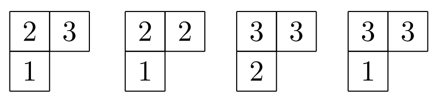
Figure 4.6: An example of $\rho(T)$, where $T \in \mathrm{SSYT}(4, 4, 2, 1)$.

In Figure 4.7 we see that $S_{(2,1)}(x_1, x_2, x_3)$ written as the sum of $QS_{(1,2)}(x_1, x_2, x_3)$ and $QS_{(2,1)}(x_1, x_2, x_3)$ by using the bijection ρ from the proof of Theorem 7. The eight composition tableaux correspond to the eight SSYT in Figure 2.6.



$x_1x_2x_3 \quad x_2^2x_3 \quad x_1^2x_2 \quad x_1^2x_3$

$$QS_{(2,1)}(x_1, x_2, x_3) = x_1x_2x_3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_3$$



$x_1x_2x_3 \quad x_1x_2^2 \quad x_2x_3^2 \quad x_1x_3^2$

$$QS_{(1,2)}(x_1, x_2, x_3) = x_1x_2x_3 + x_1x_2^2 + x_2x_3^2 + x_1x_3^2$$

Figure 4.7: An example of the quasisymmetric functions $QS_{(2,1)}(x_1, x_2, x_3)$ and $QS_{(1,2)}(x_1, x_2, x_3)$.

CHAPTER 5

THE LITTLEWOOD-RICHARDSON RULE FOR $QS_\alpha \cdot S_\lambda$

The object of this section is to develop an algorithm similar to the Remmel-Whitney rule for the product of a Schur function and a quasisymmetric Schur function. We show that the result of our algorithm is the same as that obtained by using the Littlewood-Richardson rule for quasisymmetric Schur functions.

Definition 21. Given compositions $\beta = (\beta_1, \ldots, \beta_l)$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $k \leq l$ and $\alpha_i \leq \beta_i$ for all i, define the *skew composition* $\beta//\alpha$ as the result of removing the diagram of α from the diagram of β.

For $\beta = (4, 2, 3, 2, 4)$ and $\alpha = (2, 1, 2)$, Figure 5.1 shows the diagram of the skew composition $(4, 2, 3, 2, 4)//(2, 1, 2)$.

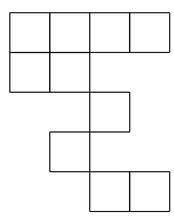


Figure 5.1: The skew diagram $(4, 2, 3, 2, 4)//(2, 1, 2)$.

Definition 22. Given compositions $\beta = (\beta_1, \ldots, \beta_l)$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $k \leq l$ and $\alpha_i \leq \beta_i$ for all i, define β/α as a skew composition $\beta//\gamma$ where γ is any weak composition with $\gamma_i \leq \beta_i$ for all i and $\gamma^+ = \alpha$, where γ^+ is γ with all 0 parts removed. There may be several distinct β/α for a given β and α depending on which weak composition is chosen.

For $\beta = (4, 2, 3, 2, 4)$ and $\alpha = (2, 1, 2)$, two diagrams of a skew composition $(4, 2, 3, 2, 4)/(2, 1, 2)$ are shown in Figure 5.2.

Definition 23. Given a filling T of a composition β, we define two configurations of cells known as *Type A* and *Type B* triples. A Type A triple is the triple of cells $T(j, k) = c, T(j, k+1) = b$,

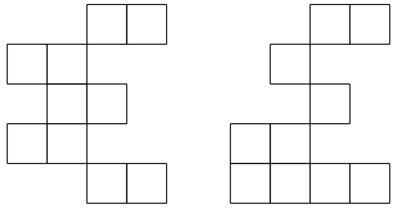
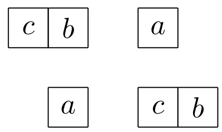


Figure 5.2: Two skew diagrams of $(4, 2, 3, 2, 4)/(2, 1, 2)$.

and $T(i, k+1) = a$, where $i < j$, $\beta_i \leq \beta_j$, and $k < \beta_i$. A Type B triple is the triple of cells $T(j, k) = a, T(i, k) = c$, and $T(i, k+1) = b$, where $i < j$, $\beta_i > \beta_j$, and $k \leq \beta_j$.



Type A Type B

Figure 5.3: The Type A and Type B triples outlined in Definition 23.

Definition 24 (Haglund et al. [3])**.** Let α, β be compositions such that $\alpha_i \leq \beta_i$ for all i. A *Littlewood-Richardson skew tableau* S of shape β/α is a filling of a diagram of β/α with additional column, called column 0, preceding the first column that is filled with 0's, where the following conditions are satisfied:

1. The cells in α are filled with 0's. If there are multiple 0's in the same column, we think of the 0's as strictly increasing from top to bottom.

2. Each row must be weakly increasing from left to right.

3. All Type A and Type B triples are *inversion triples*, meaning $a > b$ or $a < c$.

4. The *column reading word* (obtained by reading down each column starting with the right-most column, omitting all 0's), must be a lattice word.

We denote by $LRC(\beta/\alpha, \lambda)$ the set of Littlewood-Richardson composition tableaux of shape β/α with content monomial $x^{\lambda_1} x^{\lambda_2} \cdots x^{\lambda_k}$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$.

Consider when $\alpha = (2,1)$, $\beta = (3,2,3)$, and $\lambda = (3,2)$. The only Littlewood-Richardson composition tableau of shape $(3,2,3)/(2,1)$ with content $(3,2)$ can be seen in Figure 5.4.

$*$	$*$	1
$*$	1	
1	2	2

Figure 5.4: The *LRC* tableau of shape $(3,2,3)/(2,1)$ with content $(3,2)$.

Theorem 8 (Haglund et al. [3]). *Let α be a composition and λ be a partition. Then,*

$$QS_\alpha(x_1, ..., x_n) \cdot S_\lambda(x_1, ..., x_n) = \sum_\beta C^\beta_{\alpha\lambda} QS_\beta(x_1, ..., x_n)$$

where $C^\beta_{\alpha\lambda}$ is the number of Littlewood-Richardson composition tableaux (LRC) of shape β/α and has the content monomial $x^{\lambda_1} x^{\lambda_2} \cdots x^{\lambda_k}$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$.

To state the algorithm, we first need some background definitions. We then state the algorithm for multiplying $QS_\alpha \cdot S_{(1)}$, the Littlewood-Richardson rule for quasisymmetric Schur functions, and show the bijection establishing their equivalence.

Definition 25. A *reverse word composition tableau* is a composition tableau where the entries $1, \ldots, n$ are placed consecutively across rows from right to left, starting at the bottom row.

For example, Figure 5.5 shows the reverse word composition tableau for $\alpha = (3,1,3)$.

7	6	5
4		
3	2	1

Figure 5.5: Reverse word composition tableau for (3,1,3).

Definition 26. Let α be a composition of n. Define $f(\alpha)$ as the tableau obtained from the reverse word tableau T where every $x \in T$ will be replaced with $x* = n - x + 1$. If $x < y$, then $x* > y*$. Also, if $\infty* = 0$, then $0* = \infty$. Observe that $x** = x$. We call $f(\alpha)$ the standard composition filling of α.

To multiply a Schur function of shape $\lambda = (1)$ and a quasisymmetric Schur function, we need a set of rules for inserting n into $f(\alpha)$.

Definition 27. Let T be the reverse standard composition tableau of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, where $\alpha \models (n-1)$. We insert n into $f(\alpha)$.

1. Let $i = \min\{j : \alpha_j = 1\}$. Then, in $f(\alpha)$, n can be inserted in a new row between any two rows α_j, α_{j+1}, with $j < i$. Also, n may always be placed in a new row α_0 of length 1 below α_1.

2. Let $\max\{\alpha_i\} = m$. In $f(\alpha)$, if for all $j < i$, $\alpha_j \neq \alpha_i + 1$, then n can be appended to α_i.

Lemma 9. *The above rules for inserting n into a standard composition tableau will result in a valid composition tableau satisfying the Littlewood-Richardson triple condition.*

Proof. We need to show all triples fully contained in $f(\alpha)$ satisfy the triple conditions. First consider the Type A triples in $f(\alpha)$, labeled with a, b, and c as in Figure 5.3. In the reverse lexicographic filling of α, $a* < b*$. Thus, in $f(\alpha)$, $a > b$. Therefore, all Type A triples satisfy the inversion triple condition in Definition 24.

Now consider the Type B triples in $f(\alpha)$, labeled with a, b, and c as in Figure 5.3. In the reverse lexicographic filling, $a* > c*$ or $a = c = \infty$. Thus, in $f(\alpha)$, $a < c$ or $a = c = 0$. Therefore, all Type B triples satisfy the inversion triple condition in Definition 24.

Since $f(\alpha)$ satisfies the triple conditions, we need to show that the triple conditions are still preserved after inserting n.

Suppose T is a filling obtained by inserting n into $f(\alpha)$ using the condition (1) in Definition 24. Let $i = \min\{j : \alpha_j = 1\}$. Suppose a Type A triple in $f(\alpha)$ involves n. Since $\alpha_j \leq \alpha_k$ for all $k \geq j+1$, any Type A triple with n as an entry has $a = n$, so $a > b$, and the triple condition is satisfied.

Insert n into its own new row, α'_j, between rows α_j and α_{j+1} for $j < i$. Type B triples occur when $\alpha_k > \alpha'_j$ with $k \leq j$. This is true for all $k \leq j$, thus $a = n$. So, $a > b$ and the triple condition is satisfied. Therefore, all triple conditions are satisfied in T for the after insertion by condition (1).

Suppose T is a filling obtained by inserting n into $f(\alpha)$ using the second condition. Append n to the row α_i. Any Type A triples involving $\alpha_j \leq \alpha_i$ for $j < i$ has $\alpha_j < \alpha_i + 1$ and hence does not involve n. Now we need to consider all triples that include the rows above the row where n was inserted, α_i. Type A triples occur when $\alpha_k \geq \alpha_i$ with $k > i$. For all triples of this type, $a = n$. So, $a > b$ and the triple condition is satisfied.

Now consider the Type B triples for $k > i$ when $\alpha_k < \alpha_i + 1$. The only Type B triples occur when $\alpha_i = \alpha_k$ and are in the form:

$$\begin{array}{|c|c|}\hline c & b \\ \hline\end{array}$$

$$\qquad\begin{array}{|c|c|}\hline a & n \\ \hline\end{array}$$

Before insertion, there were Type A triples where $a > b$ and this satisfies the Type B inversion triple condition as well.

Consider the Type B triples for $k < i$ and $\alpha_k > \alpha_i + 1$. So, $a = n$. So, $a > b$ and the triple condition is satisfied. Therefore, all triple conditions are satisfied in T for the second condition. $\square$

To find $QS_\alpha \cdot S_{(1)}$, a rooted tree will be constructed using an algorithm similar to Algorithm 2 where each leaf is a composition of size $|\alpha| + 1$.

Definition 28. Let $Q\mathcal{O}(\alpha * (1))$ be the set of all tableaux obtained by inserting n into $f(\alpha)$. Let $Q\mathcal{O}_\beta(\alpha * (1))$ be the set of all tableaux obtained by inserting n into $f(\alpha)$ with shape β/α.

Algorithm 10. *To find the expansion of* $QS_\alpha \cdot S_{(1)}$*:*

1. *Start with the reverse word composition tableau* T *of shape* α*.*
2. *From* T*, find* $f(\alpha)$*, the reverse standard composition tableau of shape* α*. This is the root of the tree.*
3. *Do the following to* T*: Place* n *in all possible ways following the restrictions given in Definition 27. Each distinct placement creates a filling that becomes a new vertex in the tree, with an edge connected to* T*.*

4. *The leaves are the fillings in $Q\mathcal{O}(\alpha * (1))$. To obtain the summands occuring in the product $QS_\alpha \cdot S_{(1)}$, we need to replace each leaf of the tree with the quasisymmetric Schur function of the same shape. So, $QS_\alpha \cdot S_{(1)}$ is*

$$QS_\alpha \cdot S_{(1)} = \sum_{F \in Q\mathcal{O}(\alpha*(1))} QS_{sh(F)},$$

where $sh(F)$ is the shape of F.

In Figure 5.7, contains an example of the product of a Schur function of shape $\lambda = (1)$ and a quasisymmetric Schur function of shape $\alpha = (3, 1, 3)$, also denoted as $QS_{(3,1,3)} \cdot S_{(1)}$, by using Algorithm 10. We can also find the product of $QS_{(3,1,3)} \cdot S_{(1)}$ by using the Littlewood-Richardson rule. Figure 5.6 shows the Littlewood-Richardson composition tableaux.

$$\begin{array}{ccc} \ast & \ast & \ast \\ \ast & 1 & \\ \ast & \ast & \ast \end{array} \qquad \begin{array}{cccc} \ast & \ast & \ast & 1 \\ \ast & & & \\ \ast & \ast & \ast & \end{array} \qquad \begin{array}{cccc} \ast & \ast & \ast & \\ \ast & & & \\ \ast & \ast & \ast & 1 \end{array} \qquad \begin{array}{ccc} \ast & \ast & \ast \\ \ast & & \\ 1 & & \\ \ast & \ast & \ast \end{array} \qquad \begin{array}{ccc} \ast & \ast & \ast \\ \ast & & \\ \ast & \ast & \ast \\ 1 & & \end{array}$$

Figure 5.6: Littlewood-Richardson composition tableaux for $\alpha = (3, 1, 3)$ and $\lambda = (1)$.

Similar to the bijection between the tableaux resulting from the Remmel-Whitney rule and the Littlewood-Richardson tableaux for Schur functions, there is a bijection between the Littlewood-Richardson composition tableaux and the tableaux in $Q\mathcal{O}(\alpha * (1))$.

Theorem 11. *Given a composition α and partition $\lambda = (1)$, $|Q\mathcal{O}_\beta(\alpha * (1))| = C^\beta_{\alpha(1)}$, where $C^\beta_{\alpha(1)}$ is the number of Littlewood-Richardson composition tableaux of shape β/α with content* (1).

Proof. We need to show that there is a bijection $\phi : Q\mathcal{O}_\beta(\alpha * (1)) \to LRC(\beta/\alpha, (1))$. Start with a tableau $T \in Q\mathcal{O}_\beta(\alpha * (1))$ of shape β/α. Now we need to delete the entries $1, \ldots, n-1$ and replace with 0's considering 0's in the same column to be strictly increasing from top to bottom. Replace n with 1 since $\lambda = (1)$. In order for the resulting tableau to be a Littlewood-Richardson composition tableau $\phi(T) \in LRC(\beta/\alpha, (1))$, we need to make sure all triple conditions are satisfied. By Lemma 9, all triples involving n satisfy the triple conditions. Since we are replacing

n with 1, then all triples involving 1 satisfy the triple conditions. All other triples contain only 0's and are considered to satisfy the triple conditions. Therefore, $\phi(T) \in LRC(\beta/\alpha, (1))$.

Start with $F \in LRC(\beta/\alpha, (1))$. Construct $\gamma(F)$ in the following manner. First replace the 1 with n. Then place the entries $1, \ldots, n-1$ into the empty cells starting in the top row and from left to right across the row in order to create $f(\alpha)$. For the resulting tableaux to be in the set $Q\mathcal{O}_\beta(\alpha * (1))$, the triple conditions must be satisfied. Since all triples involving 1 in F satisfy the triple conditions, then all triples involving n in $\gamma(F)$ will satisfy the triple conditions when 1 is replaced with n. All other triples are in $f(\alpha)$ and satisfy the triple conditions. Therefore, $\gamma(F) \in Q\mathcal{O}_\beta(\alpha * (1))$. Observe that $\gamma(\phi(T)) = T$ for all $T \in Q\mathcal{O}_\beta(\alpha * (1))$.

Therefore, ϕ is a bijection. So the number of Littlewood-Richardson composition tableaux in the set $LRC(\beta/\alpha, (1))$, $C'^{\beta}_{\alpha(1)}$, is the same as the number of tableaux in $Q\mathcal{O}_\beta(\alpha * (1))$, where the tableaux in $Q\mathcal{O}_\beta(\alpha * (1))$ have the same shape as tableaux in $LRC(\beta/\alpha, (1))$. □

In this paper, we found an algorithm for multiplying $QS_\alpha \cdot S_{(1)}$ similar to the Remmel-Whitney rule for multiplying $S_\lambda \cdot S_\mu$. We proved that there is a bijection between the tableaux produced by the algorithm for multiplying $QS_\alpha \cdot S_{(1)}$ and the Littlewood-Richardson compositon tableaux. For further research, we want to find an algorithm for inserting larger partitions of λ and an algorithm to obtain $f(\alpha)$ by insertion.

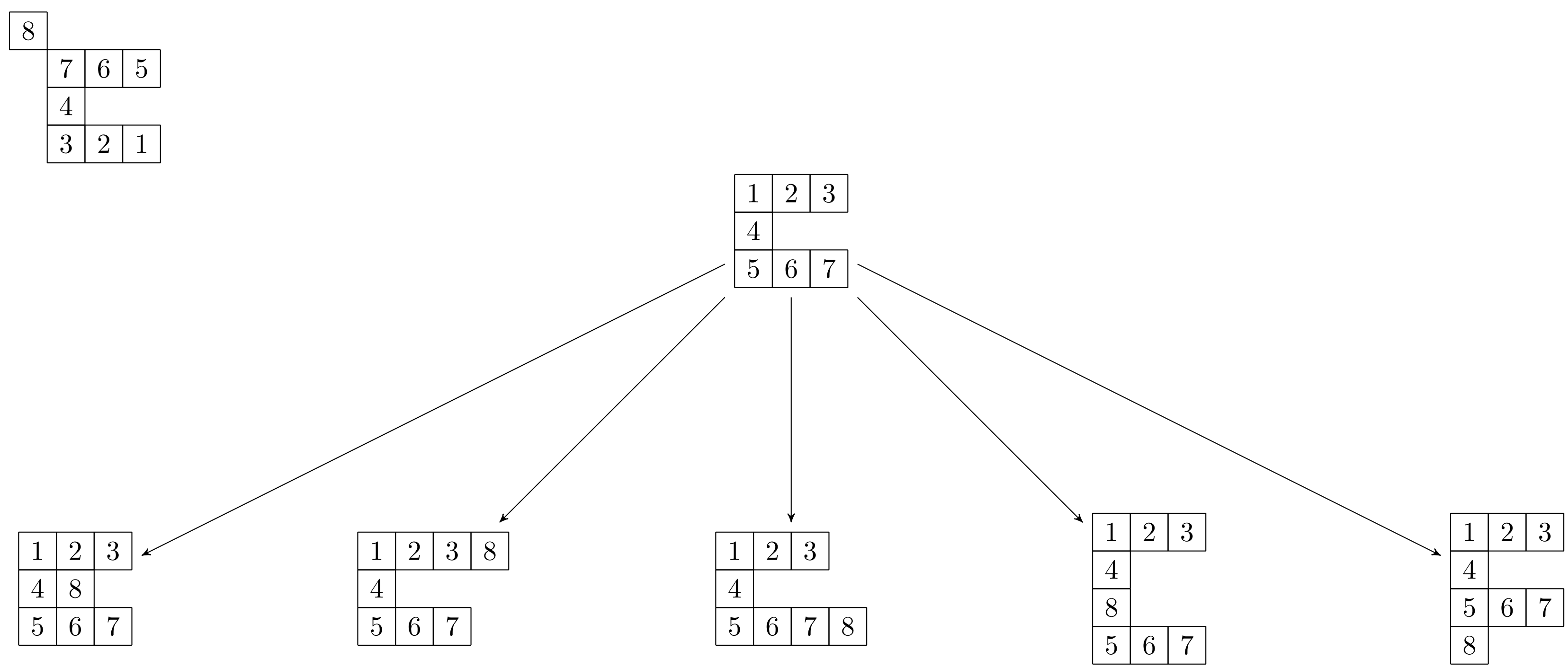


Figure 5.7: An example of multiplying a Schur function and a quasisymmetric Schur function.

$$S_{(1)} \cdot QS_{(3,1,3)} = QS_{(3,2,3)} + QS_{(4,1,3)} + QS_{(3,1,4)} + QS_{(3,1,1,3)} + QS_{(1,3,1,3)}$$

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APPENDIX A

LETTER FROM INSTITUTIONAL RESEARCH BOARD

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Office of Research Integrity
Institutional Review Board

March 4, 2015

Jennifer Marie Anderson
Department of Mathematics
Marshall University
One John Marshall Drive
Huntington, WV 25755

Dear Ms. Anderson:

This letter is in response to the submitted mathematics abstract for your thesis. After assessing the abstract it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

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Director

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Education

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- **Bachelor of Science** *summa cum laude*
  Bethany College, Bethany, WV. May 2011
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  Minor: Education 5-Adult

Research

- Graduate Research
  Marshall University, 2013-2015. Advisor Dr. Elizabeth Niese.
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    Fall 2013

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  Math I Honors, (2 sections)
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  8th grade Science, 45 min class

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